THE RESOLUTION OF ANY COLLINEATION INTO PERSPECTIVE REFLECTIONS*

BY

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The general formulæ of a collineation being

$$\rho y_i = \sum_{k=1}^{k=n} a_{ik} x_k,$$

I shall consider only collineations which have n distinct fixed points and can therefore be reduced to the *normal form*

$$\rho y_i = m_i x_i.$$

These multipliers m_i are the roots of the *characteristic equation* of the collineation

$$\begin{vmatrix} a_{11}-\rho & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\rho & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\rho & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\rho \end{vmatrix} = 0.$$

Just as any substitution can be resolved into a product of simple transpositions, so it will be shown that any collineation can be resolved into a product of perspective reflections, i. e., involutory collineations, and the minimum number of such perspective reflections will be determined. For the sake of clearness, the cases of the line, plane and ordinary space will be treated separately; the result for space of n dimensions is then deduced easily.

By *definition* a perspective reflection is one which leaves invariant a point $\kappa(\kappa_1 : \kappa_2 : \cdots : \kappa_{n+1})$ and every point in a flat space of $n-1$ dimensions, not containing κ and denoted by $a(a_1 : a_2 : \cdots : a_{n+1})$ or by the equation

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$a_x \equiv a_1 x_1 + a_2 x_2 + \cdots + a_{n+1} x_{n+1} = 0$, and which converts any point into its harmonic conjugate with respect to the point κ and the space a.

Hence the general formulæ for such a collineation may be written

$$\rho y_i = a_\kappa x_i - 2\kappa_i a_x,$$

the determinant of which is equal to $-a_\kappa^{n+1}$. By such a collineation every quadric space of $n-1$ dimensions with respect to which κ and a are pole and polar is obviously converted into itself. A perspective reflection will ordinarily be denoted by T, or by $T(a, \kappa)$ if it is desirable to put the elements a and κ in evidence.

Linear transformations of a single variable.

We can hardly use the word collineation in this case, but the general formulæ can be used, and will be used for the sake of uniformity.

THEOREM. *The general linear transformation of a single variable can be resolved into the product of two perspective reflections of period two, and in ∞^1 ways.*

If the general linear transformation S be reduced to its normal form

$$S: \qquad \rho y_i = m_i x_i$$

and be multiplied by the transformation $T(a, \kappa)$ of period two

$$T: \qquad \rho y_i = a_\kappa x_i - 2\kappa_i a_x,$$

the product $ST = U$ will be

$$U: \qquad \begin{aligned} \rho y_1 &= (-a_1\kappa_1 + a_2\kappa_2) m_1 x_1 - 2a_2\kappa_1 m_2 x_2, \\ \rho y_2 &= -2a_1\kappa_2 m_1 x_1 + (a_1\kappa_1 - a_2\kappa_2) m_2 x_2. \end{aligned}$$

This transformation U will itself be of period two if the roots of its characteristic equation

$$\rho^2 - \rho(a_1\kappa_1 - a_2\kappa_2)(m_2 - m_1) - a_\kappa^2 m_1 m_2 = 0$$

are equal and opposite, that is to say, if $\kappa_1 : \kappa_2 = a_2 : a_1$. Then T reduces to

$$y_1 : y_2 = a_2^2 x_2 : a_1^2 x_1$$

and U becomes

$$y_1 : y_2 = a_2^2 m_2 x_2 : a_1^2 m_1 x_1.$$

It is evident that U is of period two, and since $ST = U$, it follows that $S = UT$, which was to be proved. Moreover since $a_1 : a_2$ is arbitrary, excluding of course the two cases $a_1 : a_2 = 1 : 0$ and $a_1 : a_2 = 0 : 1$, the resolution can be effected in ∞^1 ways.

Collineations in the plane.

The product of two perspective reflections $T_1(a, \kappa)$ and $T_2(b, \lambda)$ will leave invariant the point μ where a and b meet and the two points on the line $(\kappa\lambda)$ which are harmonically conjugate with respect to both (a, κ) and (b, λ). If we take the triangle $(\kappa\lambda\mu)$ as coördinate triangle, the various elements involved will be

$$\kappa(1:0:0), \qquad \lambda(0:1:0),$$
$$a(a_1:a_2:0), \qquad b(b_1:b_2:0),$$

and the formulæ of the collineations T_1 and T_2 become

$$\begin{aligned} T_1: \quad & \rho y_1 = -a_1x_1 - 2a_2x_2, & T_2: \quad & \rho y_1 = b_2x_1, \\ & \rho y_2 = a_1x_2, & & \rho y_2 = -2b_1x_1 - b_2x_2, \\ & \rho y_3 = a_1x_3, & & \rho y_3 = b_2x_3. \end{aligned}$$

From this we deduce

$$\begin{aligned} T_1T_2: \quad & \rho y_1 = -a_1b_2x_1 - 2a_2b_2x_2, \\ & \rho y_2 = 2a_1b_1x_1 + (4a_2b_1 - a_1b_2)x_2, \\ & \rho y_2 = a_1b_2x_3, \end{aligned}$$

of which the characteristic equation is

$$(a_1b_2 - \rho)\left[\rho^2 - \rho(4a_2b_1 - a_1b_2) + (a_1b_2)^2\right] = 0.$$

In this equation the product of two of the roots is equal to the square of the third, $\rho_1\rho_2 = \rho_3^2$, and the collineation T_1T_2 is therefore reducible to the normal form

$$S: \qquad \rho y_1 = m_1x_1, \qquad \rho y_2 = m_2x_2, \qquad \rho y_3 = x_3,$$

where $m_1m_2 = 1$. This collineation converts into itself every conic $x_1x_2 + kx_3^2 = 0$, the point $(0:0:1)$ being the polar of the line $x_3 = 0$. Conversely:

Every plane collineation which leaves a conic invariant can be resolved into the product of two perspective reflections, and this may be effected in ∞^1 ways.

For it can be shown that every such collineation S can be reduced to the normal form S just given. With S we must compound a reflection $T_2(a, \kappa)$, where κ lies on $x_3 = 0$ and a passes through $(0:0:1)$, that is,

$$\kappa = \kappa(\kappa_1:\kappa_2:0), \qquad a = a(a_1:a_2:0), \qquad a_\kappa = a_1\kappa_1 + a_2\kappa_2.$$

The formulæ of T_2 are therefore

$$\begin{aligned} T_2: \quad & \rho y_1 = (-a_1\kappa_1 + a_2\kappa_2)x_1 - 2a_2\kappa_1x_2, \\ & \rho y_2 = -2a_1\kappa_2x_1 + (a_1\kappa_1 - a_2\kappa_2)x_2, \\ & \rho y_3 = (a_1\kappa_1 + a_2\kappa_2)x_3. \end{aligned}$$

and the characteristic equation of ST_2 is

$$(a_\kappa - \rho)[\rho^2 - \rho(a_1\kappa_1 - a_2\kappa_2)(m_2 - m_1) - a_\kappa^2] = 0.$$

Hence $ST_2 = T_1$ will itself be a perspective reflection if $a_1\kappa_1 - a_2\kappa_2 = 0$, that is, if $a_1 : a_2 = 1/\kappa_1 : 1/\kappa_2$, where $\kappa_1 : \kappa_2$ is arbitrary, only the points $1:0:0$ and $0:1:0$ being excluded. From $ST_2 = T_1$ and $T_2^2 = 1$, follows $S = T_1 T_2$, and it is clear that this resolution may be effected in ∞^1 ways.

It will now be shown that corresponding to any plane collineation U, there are ∞^3 perspective reflections T_3 such that the product $UT_3 = S$ will be of the type just considered, leaving a conic unchanged. If U be written in the normal form

$$U: \qquad \rho y_i = m_i x_i$$

and T_3 in the general form

$$T_3: \qquad \rho y_i = a_\kappa x_i - 2\kappa_i a_x,$$

the formulæ for UT_3 become

$$\rho y_1 = (a_\kappa - 2a_1\kappa_1)m_1x_1 - 2a_2\kappa_1m_2x_2 - 2a_3\kappa_1m_3x_3,$$

$$\rho y_2 = -2a_1\kappa_2m_1x_1 + (a_\kappa - 2a_2\kappa_2)m_2x_2 - 2a_3\kappa_2m_3x_3,$$

$$\rho y_3 = -2a_1\kappa_3m_1x_1 - 2a_2\kappa_3m_2x_2 + (a_\kappa - 2a_3\kappa_3)m_3x_3.$$

The characteristic equation of this collineation is

$$\begin{aligned}-\rho^3 + \rho^2[(a_\kappa - 2a_1\kappa_1)m_1 + (a_\kappa - 2a_2\kappa_2)m_2 + (a_\kappa - 2a_2\kappa_3)m_3]\\ -\rho a_\kappa[m_2m_3(a_\kappa - 2a_2\kappa_2 - 2a_3\kappa_3) + m_3m_1(a_\kappa - 2a_3\kappa_3 - 2a_1\kappa_1)\\ + m_1m_2(a_\kappa - 2a_1\kappa_1 - 2a_2\kappa_2)] - a_\kappa^3 m_1m_2m_3 = 0.\end{aligned}$$

If now UT_3 is to leave a conic unchanged, the roots of this characteristic equation must satisfy the relation $\rho_1\rho_2 = \rho_3^2$, whence

$$(\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1)^3 = \rho_1\rho_2\rho_3(\rho_1 + \rho_2 + \rho_3)^3,$$

or the coefficients of the equation satisfy the relation

$$[\sum m_2m_3(a_\kappa - 2a_2\kappa_2 - 2a_3\kappa_3)]^3 + m_1m_2m_3[\sum m_1(a_\kappa - 2a_1\kappa_1)]^3 = 0.$$

Here κ may be chosen arbitrarily, excluding only the fixed points of U, and then this condition becomes the equation of three points, through one of which a must pass. Then T_3 may be found in $\infty^2 \cdot \infty^1 = \infty^3$ ways so that $UT_3 = S$ shall be of the required type.

Combining these two results, we have the following

THEOREM. *Any plane collineation U can be resolved in ∞^4 ways into the product of three perspective reflections $T_1 T_2 T_3$.*

A case of especial interest and importance in that in which

$$\sum m_1(a_\kappa - 2a_1\kappa_1) = 0, \qquad \sum m_2 m_3(a_\kappa - 2a_2\kappa_2 - 2a_3\kappa_3) = 0,$$

for in this case $S^3 = 1$. The solution of these equations gives

$$a_1\kappa_1 : a_2\kappa_2 : a_3\kappa_3$$
$$=(m_2-m_3)(m_2m_3+m_1^2) : (m_3-m_1)(m_3m_1+m_2^2) : (m_1-m_2)(m_1m_2+m_3^2).$$

A number of well-known groups are generated by collineations S and T, where

$$S^n = 1, \qquad T^2 = 1, \qquad (ST)^3 = 1,$$

and the analysis just given readily furnishes the complete formulæ.

Collineations in space of three dimensions.

The product of two reflections in space $T_1(a, \kappa)$ and $T_2(b, \lambda)$ will leave invariant every point on the line of intersection of the planes a and b, together with two points on the line $(\kappa\lambda)$. It can be shown just as above that the normal form of the product $T_1 T_2$ will be

$$y_1 : y_2 : y_3 : y_4 = m_1 x_1 : m_2 x_2 : x_3 : x_4,$$

where $m_1 m_2 = 1$.

The product of three perspective reflections $T_1(a, \kappa)$, $T_2(b, \lambda)$ and $T_3(c, \mu)$ will leave invariant the point ν common to a, b and c, and also three points on the plane $(\kappa\lambda\mu)$. Taking $\kappa, \lambda, \mu, \nu$ as the vertices of the tetrahedron of reference, we have

$$\kappa = \kappa(1:0:0:0), \qquad \lambda = \lambda(0:1:0:0), \qquad \mu = \mu(0:0:1:0),$$
$$a = a(a_1 : a_2 : a_3 : 0), \qquad b = b(b_1 : b_2 : b_3 : 0), \qquad c = c(c_1 : c_2 : c_3 : 0),$$

and the formulæ of T_1 are simply

$$T_1: \quad \begin{aligned} \rho y_1 &= -a_1x_1 - 2a_2x_2 - 2a_3x_3, \\ \rho y_2 &= a_1x_2, \\ \rho y_3 &= a_1x_3, \\ \rho y_4 &= a_1x_4, \end{aligned}$$

with similar formulæ for T_2 and T_3. Hence we have for $T_1 T_2 T_3$ the formulæ

$$\begin{aligned} \rho y_1 &= -a_1b_2c_3x_1 - 2a_2b_2c_3x_2 - 2a_3b_2c_3x_3, \\ \rho y_2 &= 2a_1b_1c_3x_1 + (4a_2b_1c_3 - a_1b_2c_3)x_2 + (4a_3b_1c_3 - 2a_1b_3c_3)x_3, \\ \rho y_3 &= (2a_1b_2c_1 - 4a_1b_1c_2)x_1 + (4a_2b_2c_1 - 8a_2b_1c_2 + 2a_1b_2c_2)x_2 \\ &\qquad + (4a_3b_2c_1 - 8a_3b_1c_2 + 4a_1b_3c_2 - a_1b_2c_3)x_3, \\ \rho y_4 &= a_1b_2c_3x_4; \end{aligned}$$

and the characteristic equation is

$$(a_1b_2c_3-\rho)\,[-\rho^3+\rho^2(-8a_3b_1c_2+4a_1b_3c_2+4a_3b_2c_1+4a_2b_1c_3-3a_1b_2c_3)$$
$$-\rho(a_1b_2c_3)(8a_2b_3c_1-4a_1b_3c_2-4a_3b_2c_1-4a_2b_1c_3+3a_1b_2c_3)-(a_1b_2c_3)^3]=0.$$

This equation is characterized by the property that $\rho_1\rho_2\rho_3+\rho_4^3=0$, and the normal form of the collineation will be $y_1:y_2:y_3:y_4=m_1x_1:m_2x_2:m_3x_3:-x_4$, where $m_1m_2m_3=1$.

I propose now to show that any collineation S_4 in space of three dimensions can be reduced to a product S_3T_4 where the multipliers of S_3 are connected by the relation $m_1m_2m_3+m_4^3=0$ and T_4 is a reflection; that a collineation of type S_3 can be reduced to a product S_2T_3 where the multipliers of S_2 satisfy the relations $m_1m_2=m_3^2$, $m_3=m_4$, and T_3 is a reflection; finally, that a collineation of type S_2 can be reduced to a product of two reflections T_1T_2, and hence that $S_4=T_1T_2T_3T_4$.

If S_4 be reduced to the normal form

$$\rho y_i=m_ix_i$$

and T_4 be given in the general form

$$\rho y_i=a_\kappa x_i-2\kappa_i a_x,$$

then $S_4T_4=S_3$ will be of the form

$$\rho y_i=a_\kappa m_ix_i-2\kappa_i a_{mx},$$

and the characteristic equation of S_3 will be

$$\begin{vmatrix} (a_\kappa-2a_1\kappa_1)m_1-\rho & -2a_2\kappa_1m_2 & -2a_3\kappa_1m_3 & -2a_4\kappa_1m_4 \\ -2a_1\kappa_2m_1 & (a_\kappa-2a_2\kappa_2)m_2-\rho & -2a_3\kappa_2m_3 & -2a_4\kappa_2m_4 \\ -2a_1\kappa_3m_1 & -2a_2\kappa_3m_2 & (a_\kappa-2a_3\kappa_3)m_3-\rho & -2a_4\kappa_3m_4 \\ -2a_1\kappa_4m_1 & -2a_2\kappa_4m_2 & -2a_3\kappa_4m_3 & (a_\kappa-2a_4\kappa_4)m_4-\rho \end{vmatrix}=0.$$

This reduces to

$$\rho^4-\rho^3\sum(a_\kappa-2a_1\kappa_1)m_1+\rho^2a_\kappa\sum m_1m_2(a_\kappa-2a_1\kappa_1-2a_2\kappa_2)$$
$$-\rho a_\kappa^2\sum m_1m_2m_5(a_\kappa-2a_1\kappa_1-2a_2\kappa_2-2a_3\kappa_3)-a_\kappa^4m_1m_2m_3m_4=0,$$

and S_3 will be of the required character if this equation be satisfied by any fourth root of $a_\kappa^4m_1m_2m_3m_4$; T_4 can therefore be chosen in ∞^5 different ways.

In particular, if the equations

$$\sum m_1(a_\kappa-2a_1\kappa_1)=0,$$

$$\sum m_1m_2(a_\kappa-2a_1\kappa_1-2a_2\kappa_2)=0,$$

$$\sum m_1m_2m_3(a_\kappa-2a_1\kappa_1-2a_2\kappa_2-2a_3\kappa_3)=0$$

be simultaneously satisfied, $S_3^4=1$; and in this case T_4 can be chosen in ∞^3 ways.

Reducing S_3 to its normal form

$$y_1:y_2:y_3:y_4=m_1x_1:m_2x_2:m_3x_3:-x_4,$$

where $m_1m_2m_3=1$, we must choose the elements of $T_3(b,\lambda)$, so that λ lies on $x_4=0$ and b passes through $0:0:0:1$, that is to say,

$$\lambda=\lambda(\lambda_1:\lambda_2:\lambda_3:0) \qquad \text{and} \qquad b=b(b_1:b_2:b_3:0),$$

so that $b_\lambda\equiv b_1\lambda_1+b_2\lambda_2+b_3\lambda_3$. We can then write down the characteristic equation of S_3T_3 immediately as follows:

$$\begin{vmatrix} (b_\lambda-2b_1\lambda_1)m_1-\rho & -2b_2\lambda_1m_2 & -2b_3\lambda_1m_3 & 0 \\ -2b_1\lambda_2m_1 & (b_\lambda-2b_2\lambda_2)m_2-\rho & -2b_3\lambda_2m_3 & 0 \\ -2b_1\lambda_3m_1 & -2b_2\lambda_3m_2 & (b_\lambda-2b_2\lambda_2)m_3-\rho & 0 \\ 0 & 0 & 0 & -b_\lambda-\rho \end{vmatrix}=0.$$

This is equivalent to

$$-(b_\lambda+\rho)[-\rho^3+\rho^2\sum m_1(b_\lambda-2b_1\lambda_1)$$
$$-\rho b_\lambda\sum m_1m_2(b_\lambda-2b_1\lambda_1-2b_2\lambda_2)-b_\lambda^3]=0.$$

One root of this equation is obviously $-b_\lambda$; and, that the product $S_3T_3=S_2$ be of the required form, it is necessary and sufficient that a second root should also be equal to $-b_\lambda$; hence

$$\sum m_1(b_\lambda-2b_1\lambda_1)+\sum m_1m_2(b_\lambda-2b_1\lambda_1-2b_2\lambda_2)=0.$$

This reduction can evidently be performed in ∞^3 ways, and S_2 can be reduced to the normal form

$$y_1:y_2:y_3:y_4=m_1x_1:m_2x_2:x_3:x_4,$$

where $m_1m_2=1$.

In particular, if

$$\sum m_1(b_\lambda-2b_1\lambda_1)=0,$$

$$\sum m_1m_2(b_\lambda-2b_1\lambda_1-2b_2\lambda_2)=0,$$

S_2 will be of period three; and this reduction can be effected in ∞^2 ways.

The elements of $T_2(c,\mu)$ must now be so chosen that μ lies on the line $x_3=0$, $x_4=0$, and c passes through the line $x_1=0$, $x_2=0$. Then

$$\mu=\mu(\mu_1:\mu_2:0:0) \qquad \text{and} \qquad c=c(c_1:c_2:0:0),$$

so that $c_\mu = c_1\mu_1 + c_2\mu_2$. The characteristic equation of $S_2 T_2 = T_1$ is then

$$\begin{vmatrix} (c_\mu - 2c_1\mu_1)m_1 - \rho & -2c_2\mu_1 m_2 & 0 & 0 \\ -2c_1\mu_2 m_1 & (c_\mu - 2c_2\mu_2)m_2 - \rho & 0 & 0 \\ 0 & 0 & c_\mu - \rho & 0 \\ 0 & 0 & 0 & c_\mu - \rho \end{vmatrix} = 0$$

and this is equivalent to

$$(c_\mu - \rho)^2 [\rho^2 - \rho\Sigma m_1(c_\mu - 2c_1\mu_1) - c_\mu^2] = 0.$$

T_1 will be a reflection if the coefficient of ρ is equal to zero. This gives

$$c_1\mu_1 - c_2\mu_2 = 0,$$

and the reduction can be performed in ∞^1 ways.

We have then the final result:

THEOREM. *Any collineation in space may be reduced to a product of four perspective reflections in ∞^9 ways. In particular, the reduction $S_4 = T_1 T_2 T_3 T_4$, subject to the relations $(T_1 T_2)^3 = 1$, $(T_1 T_2 T_3)^4 = 1$ can be effected in ∞^6 ways.*

IV. *Collineations in space of n dimensions.*

Passing now to the general case of collineations in space of n dimensions we observe first that the product of $k+1$ reflections $(k \leqq n)$,

$$T_1(a, \kappa), \qquad T_2(b, \lambda), \qquad \cdots \qquad T_{k+1}(s, \sigma)$$

leaves invariant all the points common to the spaces $a, b, \cdots, s$, together with $k+1$ points lying in the flat space of k dimensions determined by the points $\kappa, \lambda, \cdots, \sigma$. The normal form of such a product is evidently

$$\rho y_i = m_i x_i \qquad (i = 1, 2, 3, \cdots, k+1),$$

$$\rho y_i = x_i \qquad (i = k+2, k+3, \cdots, n+1),$$

where $m_1 m_2 \cdots m_{k+1} = (-1)^{k+1}$.

Vice versa, any collineation S_{k+1} which can be reduced to the above normal form can be resolved in ∞^{k^2} ways into the product of $k+1$ perspective reflections, as I shall show immediately. We must evidently compound S_{k+1} first with a perspective reflection $T_{k+1}(s, \sigma)$ for which

$$s = s(s_1 : s_2 : \cdots s_{k+1} : 0 : 0 : \cdots 0),$$

$$\sigma = \sigma(\sigma_1 : \sigma_2 : \cdots \sigma_{k+1} : 0 : 0 : \cdots 0),$$

and hence

$$s_\sigma = s_1\sigma_1 + s_2\sigma_2 + \cdots s_{k+1}\sigma_{k+1}.$$

Then the characteristic equation of $S_{k+1}T_{k+1}$ will be

$$(s_\sigma-\rho)^{n-k}\big[(-\rho)^{k+1}+(-\rho)^k\sum m_1(s_\sigma-2s_1\sigma_1) \\ +(-\rho)^{k-1}s_\sigma\sum m_1m_2(s_\sigma-2s_1\sigma_1-2s_2\sigma_0) \\ +\cdots-m_1m_2\cdots m_{k+1}s_\sigma^{k+1}\big]=0;$$

$(n-k)$ roots of this equation are obviously $=s_\sigma$; still another root will have this value if

$$\sum m_1(s_\sigma-2s_1\sigma_1)-\sum m_1m_2(s_\sigma-2s_1\sigma_1-2s_2\sigma_2)+\cdots \\ +(-1)^{k-1}\sum m_1m_2\cdots m_k(s_\sigma-2s_1\sigma_1\cdots 2s_k\sigma_k)=0,$$

where evidently σ may be chosen arbitrarily, that is, in ∞^k ways, and s can then be chosen in ∞^{k-1} ways. The normal form of $S_{k+1}T_{k+1}=S_k$ is then of the same form as that of S_{k+1}, k being substituted for $k+1$, and the reduction can be performed in ∞^{2k-1} ways.

In particular, the collineation S_k will be of period $k+1$ if all the terms in the above equation vanish simultaneously:

$$\sum m_1(s_\sigma-2s_1\sigma_1)=0,$$
$$\sum m_1m_2(s_\sigma-2s_1\sigma_1-2s_2\sigma_2)=0,$$
etc.

In this case, there are exactly enough equations to determine the ratios

$$s_1\sigma_1:s_2\sigma_2:\cdots:s_{k+1}\sigma_{k+1},$$

so that σ may be chosen arbitrarily and s is then determined. The reduction is then possible in ∞^k ways.

Continuing this process step by step, we arrive at the general

THEOREM. *A collineation of type S_{k+1} defined by the formulæ*

$$S_{k+1}:\qquad \rho y_i=m_ix_i \qquad (i=1,2,\cdots,k+1),$$
$$\rho y_i=x_i \qquad (i=k+2,\cdots,n+1),$$

where $m_1m_2\cdots m_{k+1}=(-1)^{k+1}$ can be resolved into a product of $k+1$ perspective reflections $T_1T_2T_3\cdots T_{k+1}$ in ∞^{k^2} ways. If these reflections be subject to the conditions

$$(T_1T_2)^3=1,\qquad (T_1T_2T_3)^4=1,\text{ etc.},$$

the reduction can be effected in $\infty^{\frac{1}{2}k(k+1)}$ ways. Moreover, it is clear that $k+1$ is the minimum number of reflections involved in the reduction.

The general collineation in space of n dimensions is included in the theorem; it is only necessary to let $k=n$.